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Duchet-type theorems for powers of HHD-free graphs

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Abstract

Using an idea due to P. Duchet in proving his well-known theorem on powers of chordal graphs, we shall describe some theorems of Duchet-type for powers of graphs that have no long induced cycles. In particular, our Duchet-type theorem for HHD-free graphs improves a recent result due to Dragan, Nicolai and Brandstädt; stating that odd powers of HHD-free graphs are also HHD-free.

1. Introduction and results

For a positive integer k the kth *power* G^k is the graph with the same vertex set as G where two vertices are adjacent in G^k if and only if their distance in G is at most k. Graphs that do not contain induced cycles of length at least four are called *chordal*. In 1984, Duchet [5] proved the following classical theorem:

Theorem 1. *If G^k is chordal, then also G^{k+2}.*

Several Duchet-type results for other graph classes are known. More precisely, if $\mathscr{K}$ is the class of cocomparability graphs, of (m-)trapezoid graphs, of (unit) interval graphs, of strongly chordal graphs, of circular arc graphs, respectively, then the analogy of Theorem 1 holds (see [1, 6, 7, 11, 12]):

$$G^k \in \mathscr{K} \Rightarrow G^{k+2} \in \mathscr{K}$$

(for cocomparability graphs, (m-)trapezoid graphs, (unit) interval graphs, and strongly chordal graphs the following stronger assertion is true: $G^k \in \mathscr{K} \Rightarrow G^{k+1} \in \mathscr{K}$).

This paper will give more theorems of this kind for further graph classes; our results improve some known facts about odd powers of graphs that will be considered. First,

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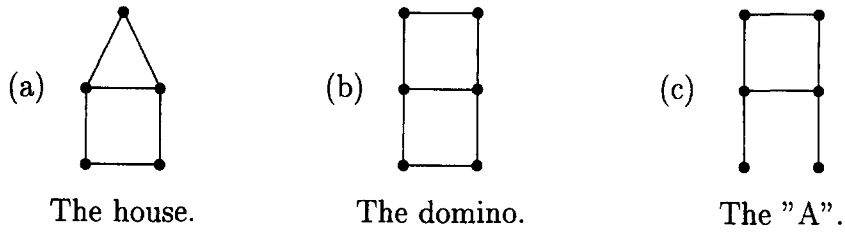


Fig. 1. The house, the domino and the "A".

the proof of Theorem 1 can be extended to obtain a slight generalization as follows. Let C_l denote the cycle of length l, $l \in \mathbb{N}$; the C_l for $l \geqslant 5$ are also called *holes*.

Theorem 2. *Let $l_0 \geqslant 4$ be a fixed integer. If G^k contains no induced C_l for all $l \geqslant l_0$, then also G^{k+2} contains no such cycle.*

This fact improves on a result in [2] saying that odd powers of graphs not containing C_l for all $l \geqslant l_0$, also do not contain C_l, $l \geqslant l_0$.

Let the house and the domino be the graphs shown in Figs. 1(a) and (b), respectively. Then a graph is said to be HHD-free if it does not contain an induced subgraph isomorphic to a hole, a house or a domino.

HHD-free graphs are introduced in [8] as an important generalization of chordal graphs. Many structural properties of this graph class are discussed in [10]. In [4] powers of HHD-free graphs are investigated; among others the following result is shown:

Theorem 3. *Odd powers of HHD-free graphs are HHD-free.*

The proof of Theorem 3 given in [4] makes use of the concept of r-dominating cliques in HHD-free graphs. By applying Duchet's proof method of Theorem 1 we can improve Theorem 3 to a result of Duchet-type:

Theorem 4. *Let G be a graph and $k \geqslant 1$ a fixed integer. If G^k is HHD-free, then so is G^{k+2}.*

This method is even more elementary and natural. The same argument yields to the following Duchet-type theorem:

Theorem 5. *Let G be a graph and $k \geqslant 1$ a fixed integer. If G^k is weak bipolarizable, then so is G^{k+2}.*

Hereby, a graph is said to be *weak bipolarizable* if and only if it is HHD-free and does not contain an induced subgraph isomorphic to an "A" (cf. Fig. 1(c)). This graph class was introduced in [9]; odd powers of weak bipolarizable graphs are discussed in [4], where it is pointed out that odd powers G^{2k+1}, $k \geqslant 1$, of weak bipolarizable graphs are chordal.

2. Proof of Theorems 2, 4 and 5

For each vertex v of a graph G, let $N(v)$ denote the neighbourhood of v, i.e. the set of all vertices in G adjacent to v. Write $N[v]$ for $N(v) \cup \{v\}$, and for a subset U of V, $N[U] := \bigcup_{u \in U} N[u]$. Let G be a graph and $V_1, \ldots, V_m$ be (not necessarily different) subsets of $V(G)$. Then the graph $G(V_1, \ldots, V_m)$ is defined as follows:

— The vertices are the sets $V_1, \ldots, V_m$.
— For $i, j \in \{1, \ldots, m\}$, $i \neq j$, V_i and V_j are adjacent in $G(V_1, \ldots, V_m)$ if and only if $N_G[V_i] \cap V_j \neq \emptyset$.

For example, if $G = (\{1, 2, 3, 4\}, \{12, 23, 34, 14, 13\})$, and $V_1 = V_3 = N[1] = N[3]$, $V_2 = N(2)$, then $G(V_1, V_2, V_3)$ is a complete graph with three vertices V_1, V_2 and V_3. The meaning of this construction for powers of graphs is captured in the following simple, but useful fact due to Duchet [5]:

Observation 6. *Let G be a graph with vertices $v_1, \ldots, v_n$. Then the graphs G^{k+2} and $(G^k)(N_G[v_1], \ldots, N_G[v_n])$ are isomorphic.*

In the following $G = (V, E)$ is a graph and $V_1, \ldots, V_m$ are some subsets of V such that for each $i \in \{1, \ldots, m\}$ the subgraph induced by V_i is connected. We often write $G' = (V', E')$ for the graph $G(V_1, \ldots, V_m)$. The following three lemmata together with Observation 6 imply the Theorems 2, 4 and 5, directly.

Lemma 7. *Let $l_0 \geqslant 4$ be an integer. If G has no induced C_l for all $l \geqslant l_0$, then so is $G(V_1, \ldots, V_m)$.*

Lemma 8. *If G is HHD-free, then so is $G(V_1, \ldots, V_m)$.*

Lemma 9. *If G is weak bipolarizable, then so is $G(V_1, \ldots, V_m)$.*

For the proof of the above lemmata we need some definitions on walks. Let $v_1, \ldots, v_k$ be (not necessarily different) vertices of G. Then we call the k-tuple $(v_1, \ldots, v_k)$ a *walk* of *length* $k-1$ if, for all $i \in \{1, \ldots, k-1\}$, $v_i v_{i+1} \in E(G)$. For $(v_1, \ldots, v_k)$ we shortly write $v_1, \ldots, v_k$. If the vertices $v_1, \ldots, v_k$ are pairwise different then we call the walk $v_1, \ldots, v_k$ a *path*.

Let $A = A_1 A_2 \ldots A_l A_1$, be an induced cycle in G'. For vertices $v_1, \ldots, v_p \in V(G)$ a closed walk $\pi = v_1 \ldots v_p v_1$ (in G) is called a *closed A-walk* of *length* p if and only if there exist integers $i_1^{(1)}, i_1^{(2)}, i_2^{(1)}, i_2^{(2)}, \ldots, i_l^{(1)}, i_l^{(2)}$, which fulfil the following three properties:

(i) $1 = i_1^{(1)} \leqslant i_1^{(2)} \leqslant i_2^{(1)} \leqslant i_2^{(2)} \leqslant \cdots \leqslant i_l^{(1)} \leqslant i_l^{(2)} = p$,
(ii) $i_{\nu+1}^{(1)} - i_\nu^{(2)} \leqslant 1$ for all $\nu \in \{1, \ldots, l-1\}$,
(iii) $\{v_i \colon i_\nu^{(1)} \leqslant i \leqslant i_\nu^{(2)}\} \subseteq A_\nu$ for all $\nu \in \{1, \ldots, l\}$.

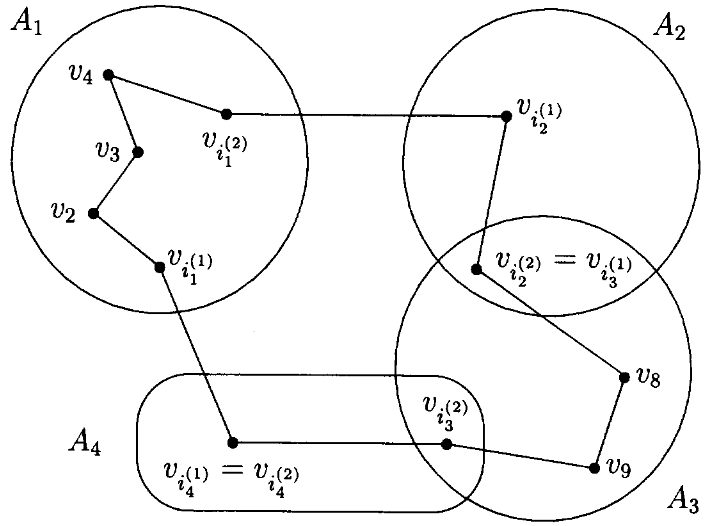


Fig. 2. An example to the A-walk definition; π is here a path.

We call the tuple $(i_1^{(1)}, i_1^{(2)}, i_2^{(1)}, i_2^{(2)}, \ldots, i_l^{(1)}, i_l^{(2)})$ an *A-partition* of π. An example to this definition is given in Fig. 2. π is called a *minimum closed A-walk* if there does not exist a closed A-walk with length less than p.

Observation 10. *For every induced cycle A in G' there exists a closed A-walk.*

Proof. Since the graphs induced by A_k and $A_k \cup A_{k+1}$ are connected, the assertion follows by induction. $\square$

Lemma 7 is a direct consequence of the next, more general observation.

Observation 11. *Let $l \geqslant 4$ be an integer and A an induced cycle of length l in G'. Then each minimum closed A-walk is an induced cycle of length at least l in G.*

Proof. Let $A = A_0 A_1 \ldots A_{l-1} A_0$ be an induced cycle of length l in G'. Further let $\pi = v_0 v_1 \ldots v_p v_0$ an arbitrary minimum closed A-walk with A-partition $(i_0^{(1)}, i_0^{(2)}, i_1^{(1)}, i_1^{(2)}, \ldots, i_{l-1}^{(1)}, i_{l-1}^{(2)})$. Then the observation follows directly from the next three claims.

Claim 1. *If $k \in \{0, \ldots, l-1\}$ then $(A_k \backslash (A_{k-1} \cup A_{k+1})) \cap \{v_0, \ldots, v_p\} \neq \emptyset$ (index addition modulo l). In particular, $p \geqslant l - 1$.*

Proof. Suppose to the contrary that there exists a $k \in \{0, \ldots, l-1\}$ with $(A_k \backslash (A_{k-1} \cup A_{k+1})) \cap \{v_0, \ldots, v_p\} = \emptyset$. Since A is an induced cycle in G' it follows $\{v_i: i_k^{(1)} \leqslant i \leqslant i_k^{(2)}\} \subseteq A_{k-1} \cup A_{k+1}$. Therefore, due to the definition of a closed A-walk, A_{k-1} and A_{k+1} are adjacent in G', in contradiction to $l \geqslant 4$. $\square$

Claim 2. *If $i, j \in \{0, \dots, p\}$, $i \neq j$, then $v_i \neq v_j$*, i.e. *π is a cycle.*

Proof. Suppose $v_i = v_j$ for some $i, j \in \{0, \dots, p\}$, $i < j$. In the following we show that

$$\pi' := v_0 v_1 \dots v_i v_{j+1} \dots v_p v_0$$

is a closed A-walk, in contradiction to the minimality of π. If $v_i \in A_k \backslash (A_{k-1} \cup A_{k+1})$ for some $k \in \{0, \dots, l-1\}$ this is clearly the case since $i_k^{(1)} \leqslant i < j \leqslant i_k^{(2)}$. Therefore, let $v_i \in A_{k-1} \cap A_k$ for some $k \in \{0, \dots, l-1\}$. Then $i_{k-1}^{(1)} < i < j < i_k^{(2)}$. Let W'_{k-1} (resp. W'_k) be the walk W_{k-1} (resp. W_k) without the vertices, whose indices are in $\{i+1, i+2, \dots, j\}$ ($W_\nu = v_{i_\nu^{(1)}} \dots v_{i_\nu^{(2)}}$, $\nu = 0, \dots, l-1$). Then it is easy to see that the corresponding indices of the endvertices of $W_1, \dots, W_{k-2}, W'_{k-1}, W'_k, W_{k+1}, \dots, W_{l-1}$ form an A-partition of π'. □

The minimality of the A-walk π implies immediately

Claim 3. *π has no chords.* □

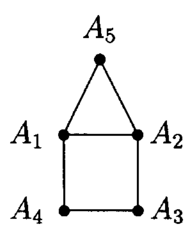
Observation 12. *Let G be a hole-free graph and $A = A_0 A_1 A_2 A_3 A_0$ an induced cycle in G'. Then for $i = 0, 1, 2, 3$ there exist vertices $v_i \in A_i \backslash (A_{i-1} \cup A_{i+1})$ (index addition modulo 4) such that $v_0 v_1 v_2 v_3 v_0$ is an induced cycle in G (see Fig. 3). Furthermore, all minimum closed A-walks are of this form.*

Proof. Since G is hole-free by Observation 11 each minimum closed A-walk is an induced cycle of length four. The structure follows immediately by Claim 1 in the proof of Observation 11. □

For a path π, let $V(\pi)$ (resp. $E(\pi)$) be the vertex (resp. edge) set of π. Now we can state the proof of Lemma 8:

Proof. By Lemma 7 we have only to show that G' contains no induced subgraph isomorphic to a house or a domino.

First, we suppose that G' contains the induced house



Let $A := A_1 A_2 A_3 A_4 A_1$ and $\mathscr{A}$ be the set of all minimum closed A-walks. By Observation 12 it is guaranteed that $\mathscr{A} \neq \emptyset$. For each $\mathfrak{w} = w_1 w_2 w_3 w_4 w_1 \in \mathscr{A}$ we

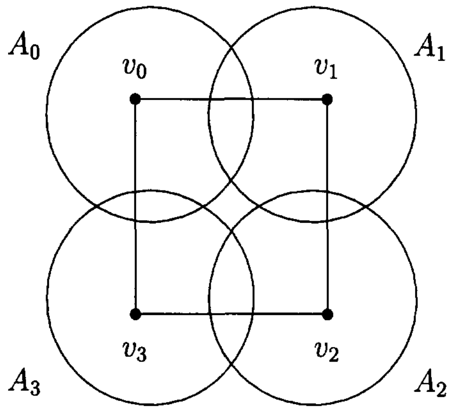


Fig. 3. For Observation 12. The intersections are possibly empty.

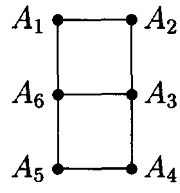
consider a shortest path $\pi_{\mathfrak{w}}$ in $A_1 \cup A_5$ from w_1 to a vertex $x_{\mathfrak{w}}$ of A_5 (since $A_1A_5 \in E(G')$ such a path must exist). Further let $\pi'_{\mathfrak{w}}$ be a shortest path in $A_5 \cup A_2$ from $x_{\mathfrak{w}}$ to w_2. Now, let $\mathfrak{v} = v_1v_2v_3v_4v_1$ an element of $\mathscr{A}$ with minimal value $|V(\pi_{\mathfrak{v}})| + |V(\pi'_{\mathfrak{v}})|$.

We claim that no vertex of $V(\pi_{\mathfrak{v}})\backslash\{v_1\}$ is adjacent to v_4. We suppose the contrary. Let $\pi_{\mathfrak{v}} =: v_1y_1 \ldots y_kx_{\mathfrak{v}}$. Since G is hole-free, the vertex v_4 is adjacent to y_1 or y_2. If $v_4y_1 \in E$ we conclude $y_1v_2 \in E$ since G is house-free. But then $\mathfrak{y} := y_1v_2v_3v_4y_1$ is an element of $\mathscr{A}$ with $|V(\pi_{\mathfrak{y}})| + |V(\pi_{\mathfrak{y}})'| < |V(\pi_{\mathfrak{v}})| + |V(\pi'_{\mathfrak{v}})|$, in contradiction to the minimality of $\mathfrak{v}$. Now we consider the case $y_1v_4 \notin E$, $y_2v_4 \in E$. If $y_1v_2 \in E$ then due to $y_1v_4 \notin E$, $y_1v_3 \notin E$ $\{v_1, v_2, v_3, v_4, y_1\}$ induces a house in G — a contradiction. Thus, $y_1v_2 \notin E$. Since $\{v_1, v_2, v_3, v_4, y_1, y_2\}$ cannot induce a domino in G, we conclude $y_2v_2 \in E$. Then, analogously to the case $v_4y_1 \in E$, $y_2v_2v_3v_4y_2$ is an element of $\mathscr{A}$ contradicting the minimality of $\mathfrak{v}$.

With the same arguments one can show that no vertex of $V(\pi'_{\mathfrak{v}})\backslash\{v_2\}$ is adjacent to v_3.

Now, since $|E(\pi_{\mathfrak{v}})|$, $|E(\pi'_{\mathfrak{v}})| \geqslant 1$, $\pi_{\mathfrak{v}} \cap \{v_2, v_3, v_4\} = \emptyset$, $\pi'_{\mathfrak{v}} \cap \{v_1, v_3, v_4\} = \emptyset$ there exists an induced cycle $v_1v_2z_1 \ldots z_lv_1$ with $z_1, \ldots, z_l \in (V(\pi_{\mathfrak{v}}) \cup V(\pi'_{\mathfrak{v}}))\backslash\{v_1, v_2\}$, $l \geqslant 1$. Since G is hole-free we conclude $l = 1$ or $l = 2$. In the first case $\{v_1, v_2, v_3, v_4, z_1\}$ induces a house and in the latter case $\{v_1, v_2, v_3, v_4, z_1, z_2\}$ induces a domino.

Now, we suppose that G' contains the induced domino



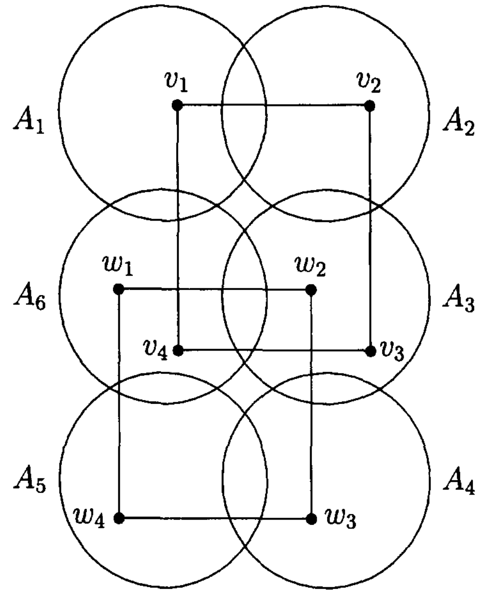


Fig. 4. For the proof of Lemma 8.

Let $v_1v_2v_3v_4v_1$ a minimum closed $A_1A_2A_3A_6$-walk and $w_1w_2w_3w_4w_1$ a minimum closed $A_6A_3A_4A_5$-walk such that $d_{A_6}(w_1, v_4) + d_{A_3}(w_2, v_3)$ is minimal (cf. Fig. 4). Hereby, for $U \subseteq V$ and vertices $a, b \in U$ the value $d_U(a, b)$ denotes the distance of the vertices a, b in the subgraph induced by U.

With π_1 we denote a shortest path in G_{A_6} connecting v_4 and w_1, with π_2 we denote a shortest path in G_{A_3} connecting v_3 and w_2. (Note: π_1 and π_2 are not necessarily disjoint.) In the same way as above one can show the following properties:

— no vertex of $V(\pi_1)\backslash\{w_1\}$ is adjacent to w_4,

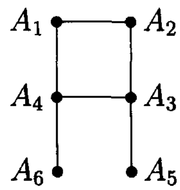
— no vertex of $V(\pi_2)\backslash\{w_2\}$ is adjacent to w_3,

— no vertex of $V(\pi_1)\backslash\{v_4\}$ is adjacent to v_1 and

— no vertex of $V(\pi_2)\backslash\{v_3\}$ is adjacent to v_2.

So, if $|E(\pi_1)| + |E(\pi_2)| > 0$ we get an induced house or domino in G. Otherwise, the vertices $v_1, \ldots, v_4, w_1, \ldots, w_4$ induce a domino. □

Proof of Lemma 9. By Lemma 8 the graph G' is HHD-free. Suppose that G' contains the induced "A"



Let $A := A_1A_2A_3A_4A_1$ and $\mathscr{A}$ be the set of all minimum closed A-walks. For each $\mathfrak{w} = w_1w_2w_3w_4w_1 \in \mathscr{A}$ we consider a shortest path $\pi_{\mathfrak{w}}$ in $A_4 \cup A_6$ from w_4 to a vertex of A_6 (since, $A_4A_6 \in E(G')$ such a path must exist) and a shortest path $\pi'_{\mathfrak{w}}$ in $A_3 \cup A_5$ from w_3 to a vertex of A_5. Now, let $\mathfrak{v} = v_1v_2v_3v_4v_1$ an element of $\mathscr{A}$ with minimum value $|V(\pi_{\mathfrak{v}})| + |V(\pi'_{\mathfrak{v}})|$.

As in the proof of Lemma 8 we get the following properties:

— no vertex of $V(\pi_{\mathfrak{v}})\backslash\{v_4\}$ is adjacent to v_1,

— no vertex of $V(\pi'_{\mathfrak{v}})\backslash\{v_3\}$ is adjacent to v_2.

Further, we conclude $E(\pi_{\mathfrak{v}}),\ E(\pi'_{\mathfrak{v}}) \geqslant 1$, because $A_1A_6, A_2A_5 \notin E(G')$. Let x be the neighbour of v_4 in $\pi_{\mathfrak{v}}$ and y be the neighbour of v_3 in $\pi'_{\mathfrak{v}}$. Since G is house-free we get $x \neq y$, xv_3, $yv_4 \notin E(G)$. So either $\{v_1, v_2, v_3, v_4, x, y\}$ induces a domino or an "A". □

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